Name: ______________________________

Teacher: ____________________________

Class: ______________________________

FORT STREET HIGH SCHOOL

2008

HIGHER SCHOOL CERTIFICATE COURSE

Assessment Task 3

Mathematics Extension 1

Time allowed: 1 hour

Outcomes Assessed	Questions
Determines integrals by reduction to a standard form through a given substitution.	1
Uses the relationship between functions, inverse functions, their derivatives and integrals.	2, 3
Synthesises mathematical solutions to harder problems and communicates them in appropriate form.	4, 5

Question	**1**	**2**	**3**	**4**	**5**	**Total**	**%**
Marks	**/11**	**/11**	**/11**	**/11**	**/11**	**/55**	

Directions to candidates:

- Attempt all questions
- The marks allocated for each question are indicated
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Board – approved calculators may be used
- Each new question is to be started on a new page

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \, \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Total marks – 55
Attempt Questions 1 – 5
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 1. (11 marks)

a) Find $\int \frac{e^x}{\sqrt{1-e^{2x}}}\,dx$ using the substation $u = e^x$ **3**

b) Evaluate $\int_3^4 x\sqrt{4-x}\,dx$ using the substitution $x = 4-u$ **3**

c) i) Express $\cos 2x$ in terms of $\cos x$. **1**

ii) Hence using the substitution $x = \sin^2\theta$, evaluate $\int_0^1 \sqrt{\frac{1-x}{x}}\,dx$ **4**

Question 2. (11 marks)

a) i) A function is defined by $f(x) = |x-2|$.

Explain why $f(x)$ does not have an inverse function. **1**

ii) State the largest positive domain which will define the inverse function $f^{-1}(x)$. **1**

iii) Find $f^{-1}(x)$ in terms of x and state its domain and range. **3**

b) i) State the domain and range of $f(x) = \frac{1}{2}\cos^{-1}(x-1)$. **2**

ii) Hence sketch the graph of $f(x)$ clearly showing coordinates of end points. **2**

c) Find the exact value of $\sin^{-1}\frac{\sqrt{3}}{2} + \cos^{-1}\frac{1}{\sqrt{2}} - \sin^{-1}\left(\sin\frac{5\pi}{6}\right)$ **2**

Question 3. (11 marks)

a) Differentiate i) $\sin^{-1}\left(\frac{3x}{2}\right)$ **2**

ii) $\cos^{-1}(5x)$ **2**

iii) $\tan^{-1}\left(\tan\frac{x}{2}\right)$ **3**

b) Find the primitive of $\frac{1}{9+4x^2}$ **2**

c) Find $\int \left(4-x^2\right)^{-\frac{1}{2}}\ dx$ **2**

Question 4. (11 marks)

a) i) Write x^2+4x+5 in the form $(x+a)^2+b^2$ **1**

ii) Hence evaluate $\int_{-3}^{-1} \frac{dx}{x^2+4x+5}$ **3**

b) i) Differentiate $\sin^{-1} x+\sqrt{1-x^2}$ **3**

ii) Hence evaluate $\int_0^{\frac{1}{2}} \sqrt{\frac{1-x}{1+x}}\, dx$ **4**

Question 5. **(11 marks)**

a) Evaluate $\int_0^3 \frac{1}{3+x^2}\, dx$ **3**

b) i) Show the curves $y=\cos^{-1}x$ and $y=2\tan^{-1}(1-x)$ intersect the y axis at the same point. **3**

ii) Hence, or otherwise, show there is a common tangent at this point. **2**

c) A region is bounded by the curve $y=\sin^{-1}x$, the $x-$ axis and the line $x=1$. **3**

Use Simpson's rule with 3 function values to find the approximation to the area of the region correct to two decimal places.

Extension 1, Task 3 2008.

Question 1 (11marks)

a) $\int \frac{e^x}{\sqrt{1-e^{2x}}}\,dx$, $u = e^x$, $du = e^x dx$.

$\int e^x.(1-e^{2x})^{-\frac{1}{2}}\,dx$

$\int \frac{du}{\sqrt{1-u^2}}$

$\sin^{-1} u + c$

$\sin^{-1} e^x + c$

Generally well done

b) $\int_3^4 x\sqrt{4-x}\,dx$.

$x = 4-u$, $u = 4-x$
$dx = -du$.
if $x=3$, $u=1$
$x=4$, $u=0$.

$-\int_1^0 (4-u)\cdot u^{\frac{1}{2}}\,du$

$\int_0^1 4u^{\frac{1}{2}} - u^{\frac{3}{2}}\,du$

$\left.\frac{2.4u^{\frac{3}{2}}}{3} - \frac{2u^{\frac{5}{2}}}{5}\right]_0^1$

$\frac{2.4.1^{\frac{3}{2}}}{3} - \frac{2.1^{\frac{5}{2}}}{5} - (0-0)$

$\frac{8}{3} - \frac{2}{5}$

$\frac{34}{15}$

Some students did not find limits in terms of u.

c) i) $\cos 2x = 2\cos^2 x - 1$ (1)

ii) $x = \sin^2\theta$

$dx = 2\sin\theta\cos\theta\,d\theta$.

if $x=0$, $\sin^2\theta = 0$
$\theta = 0$

$x=1$, $\sin^2\theta = 1$
$\theta = \frac{\pi}{2}$

$\int_0^1 \sqrt{\frac{1-x}{x}}\,dx = \int_0^{\frac{\pi}{2}} \sqrt{\frac{1-\sin^2\theta}{\sin^2\theta}}\,.\,2\sin\theta\cos\theta\,d\theta$.

$= \int_0^{\frac{\pi}{2}} \sqrt{\frac{\cos^2\theta}{\sin^2\theta}}\,.\,2\sin\theta\cos\theta\,d\theta$

$= \int_0^{\frac{\pi}{2}} \frac{\cos\theta}{\cancel{\sin\theta}}\,.\,2\cancel{\sin\theta}\cos\theta\,d\theta$

$= \int_0^{\frac{\pi}{2}} 2\cos^2\theta\,d\theta$

$= \int_0^{\frac{\pi}{2}} (\cos 2\theta + 1)\,d\theta$

$= \left.\frac{\sin 2\theta}{2} + \theta\right]_0^{\frac{\pi}{2}}$

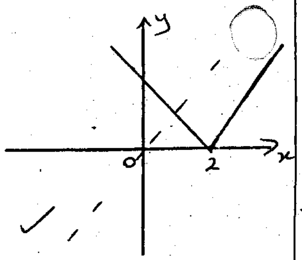
$= \frac{\sin\pi}{2} + \frac{\pi}{2} - \left(\frac{\sin 0}{2} + 0\right)$

$= \frac{\pi}{2}$

Some students did not find the relations between dx + $d\theta$.

Question 2 (11 marks).

a) i) $f(x) = |x-2|$



f(x) does not have an inverse function because it cannot be reflected in the line $y = x$ to give one to one correspondence ✓

Some students did not clearly explain why the inverse doesn't exist.

ii) $x \geq 2$ will define the function ✓

iii) $y = |x-2|$

$x = y - 2$

$x + 2 = y$

$\therefore f^{-1}(x) = x + 2$. ✓

Domain: $\{x : x \geq 0\}$ ✓

Range: $\{y : y \geq 2\}$ ✓

Many students did not know how to interchan the x + y to get $f^{-1}(x)$.

$y \neq -x + 2$ as $x \geq 2$.

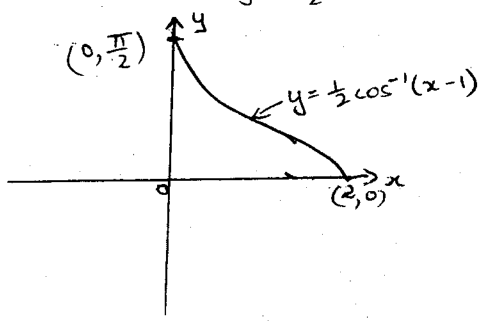
b) i) $f(x) = \frac{1}{2}\cos^{-1}(x-1)$

Domain: $-1 \leq x - 1 \leq 1$

$0 \leq x \leq 2$ ✓

Range: $0 \leq y \leq \pi$

$0 \leq \frac{1}{2}y \leq \frac{\pi}{2}$ ✓

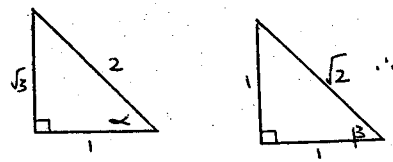


✓ ✓

Good but Domain + range need to be clearly state which is which.

c) $\sin^{-1}\frac{\sqrt{3}}{2}$ $\cos^{-1}\frac{1}{\sqrt{2}} - \sin^{-1}(\sin\frac{5\pi}{6})$

Let $\alpha = \sin^{-1}\frac{\sqrt{3}}{2}$, $\beta = \cos^{-1}\frac{1}{\sqrt{2}}$



$\therefore \alpha = \frac{\pi}{3}$

$\beta = \frac{\pi}{4}$

and $\sin^{-1}(\sin\frac{5\pi}{6}) = \frac{\pi}{6}$ ✓

so $\sin^{-1}\frac{\sqrt{3}}{2} + \cos^{-1}\frac{1}{\sqrt{2}} - \sin^{-1}(\sin\frac{5\pi}{6})$

$\frac{\pi}{3} + \frac{\pi}{4} - \frac{\pi}{6}$

$\frac{4\pi + 3\pi - 2\pi}{12}$

$\frac{5\pi}{12}$ ✓

Evaluation of $\sin^{-1}(\sin\frac{5\pi}{6})$ caused problems for some studen

$\sin^{-1}(\sin\frac{5\pi}{6}) \neq \frac{5\pi}{6}$

since

$-\frac{\pi}{2} < y < \frac{\pi}{2}$

Question 3 (11 marks)

a) i) $\frac{d}{dx}\sin^{-1}\left(\frac{3x}{2}\right) = \frac{3}{\sqrt{2^2-(3x)^2}}$ ✓

$= \frac{3}{\sqrt{4-9x^2}}$ ✓

Incorrect manipulation of a^2 and x^2

ii) $\frac{d}{dx}\cos^{-1}5x = \frac{-1}{\sqrt{1-(5x)^2}} \times 5$ ✓

$= \frac{-5}{\sqrt{1-25x^2}}$ ✓

Mostly well done

iii) $\frac{d}{dx}\tan^{-1}\left(\tan\frac{x}{2}\right)$, let $u = \tan\frac{x}{2}$

$du = \frac{1}{2}\sec^2\frac{x}{2}$ ✓

$\frac{1}{1+\left(\tan\frac{x}{2}\right)^2} \times \frac{1}{2}\sec^2\frac{x}{2}$

$\frac{1}{\sec^2\frac{x}{2}} \times \frac{1}{2}\sec^2\frac{x}{2}$ ✓

$\frac{1}{2}$ ✓

Alternate method

$\frac{d}{dx}\tan^{-1}\left(\tan\frac{x}{2}\right)$ ✓

$\frac{d}{dx}\frac{x}{2}$ ✓ since $\tan^{-1}(\tan x) = x$

$\frac{1}{2}$ ✓

Many students did not correctly differentiate $\tan\frac{x}{2}$.

Others failed to recognise that $1+\left(\tan\frac{x}{2}\right)^2 = \sec^2\frac{x}{2}$

b) $f'(x) = \frac{1}{9+4x^2}$

$f(x) = \frac{1}{4}\int \frac{1}{\frac{9}{4}+x^2}\,dx$ ✓

$= \frac{1}{4}\times\frac{2}{3}\int\frac{\frac{3}{2}}{\left(\frac{3}{2}\right)^2+x^2}\,dx = \frac{1}{6}\tan^{-1}\frac{2x}{3}+c$ ✓

Many students had difficulty finding correct co-eff. of $\tan^{-1}\frac{2x}{3}$

c) $\int (4-x^2)^{-\frac{1}{2}}\,dx = \int\frac{1}{\sqrt{4-x^2}}\,dx$ ✓

$= \sin^{-1}\frac{x}{2}+c$ ✓

Well done by those students who recognised $(4-x^2)^{-\frac{1}{2}} = \frac{1}{\sqrt{4-x^2}}$

Question 4 (11 marks)

a) i) $x^2+4x+2^2+5-2^2$

$(x+2)^2+1$ ✓

Usually well done

ii) $\int_{-3}^{-1}\frac{dx}{x^2+4x+5}$

$\int_{-3}^{-1}\frac{dx}{(x+2)^2+1}$ ✓

$\left.\tan^{-1}(x+2)\right]_{-3}^{-1}$ ✓

$\tan^{-1}(-1+2) - \tan^{-1}(-3+2)$

$\tan^{-1}1 - \tan^{-1}(-1)$

$\frac{\pi}{4} + \frac{\pi}{4}$

$\frac{\pi}{2}$ ✓

Some student had difficulty with the substitution int inverse tan. so values such as $\tan^{-1}(-3)$ were difficult work with.

b) i) $\frac{d}{dx}\left(\sin^{-1}x+\sqrt{1-x^2}\right)$

$\frac{d}{dx}\left[\sin^{-1}x+(1-x^2)^{\frac{1}{2}}\right]$ ✓

$\frac{1}{\sqrt{1-x^2}} + \frac{1}{2}(1-x^2)^{-\frac{1}{2}}\times -2x$ ✓

$\frac{1}{\sqrt{1-x^2}} - \frac{x}{\sqrt{1-x^2}}$

$\frac{1-x}{\sqrt{1-x^2}}$

$\frac{1-x}{\sqrt{(1-x)(1+x)}}$

$\sqrt{\frac{1-x}{1+x}}$ ✓

Incorrect use of chain rule particularly use of minus sign

This mark was also given if the simplification occured in ii)

ii) $\displaystyle\int_0^{\frac{1}{2}} \sqrt{\frac{1-x}{1+x}}\,dx$

$\left[\sin^{-1}x + \sqrt{1-x^2}\right]_0^{\frac{1}{2}}$

$\sin^{-1}(\tfrac{1}{2}) + \sqrt{1-\tfrac{1}{4}} - \sin^{-1}0 - \sqrt{1}$

$\frac{\pi}{6} + \frac{\sqrt{3}}{2} - 1$

The link between the derivative in part i) was missed because derivative was not simplified sufficently.

"Hence" was ignored by some students.

Question 5 (11 marks)

a) $\displaystyle\int_0^3 \frac{1}{3+x^2}\,dx = \left[\frac{1}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}}\right]_0^3$

$= \frac{1}{\sqrt{3}}\left[\tan^{-1}\frac{3}{\sqrt{3}} - \tan^{-1}0\right]$

$= \frac{1}{\sqrt{3}} \times \frac{\pi}{3}$

$= \frac{\pi\sqrt{3}}{9}$

Mostly well executed

Some students did not use the table of standard integrals, so omitting $\frac{1}{\sqrt{3}}$.

b) i) $y = \cos^{-1}x$ and $y = 2\tan^{-1}(1-x)$

At the y axis $x=0$

so $y = \frac{\pi}{2}$ and $y = 2 \times \frac{\pi}{4}$
$= \frac{\pi}{2}$

$\therefore$ curves intersect at $(0, \frac{\pi}{2})$ on the y axis

Some students attempted to find x intercept instead of y.

ii) $y = \cos^{-1}x$ and $y = 2\tan^{-1}(1-x)$

$y' = \frac{-1}{\sqrt{1-x^2}}$ and $y' = \frac{-2}{1+(1-x)^2}$

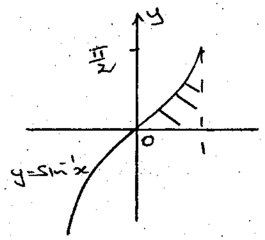
when $x = 0$, $y' = -1$; $y' = \frac{-2}{1+1}$
$= -1$

$\therefore$ tangents have the same

Inaccuracies with the sign of y' impeded some students.

c)

x	0	$\frac{1}{2}$	1
$f(x)$	0	$\frac{\pi}{6}$	$\frac{\pi}{2}$



$A \doteqdot \displaystyle\int_0^1 f(x)\,dx$

$= \frac{b-a}{6}\left[f(a) + f(b) + 4f\left(\frac{a+b}{2}\right)\right]$

$= \frac{1}{6}\left[0 + \frac{\pi}{2} + 4 \times \frac{\pi}{6}\right]$

$= \frac{7\pi}{36}$

$\doteqdot 0.610865238$

$= 0.61$ (to 2d.p)

area is approx $0.61\ u^2$.

Poorly done by a no. of students

- some did not know rule
- others used incorrect no. of function values
- h incorrect
- did not use radian measure
- did not round off value correctl